# Some properties of generalized connections in quantum gravity

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**Abstract**

The quantum completion $\bar{\mathcal{A}}$ of the space of connections in a manifold can be seen as the set of all morphisms from the groupoid of the edges of the manifold to the (compact) gauge group. This algebraic construction generalizes the analogous description of the gauge-invariant quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ of Ashtekar and Isham, clarifying the relation between the two spaces. Using this setup, we present a characterization of $\bar{\mathcal{A}}$ which brings the gauge-invariant degrees of freedom to the foreground, thus making the action of the gauge group more transparent.

## 1 Introduction

Theories of connections play an important role in the description of fundamental interactions, including Yang-Mills theories and gravity in the Ashtekar formulation [1, 2]. Typically in such cases, the classical configuration space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations is an infinite dimensional non-linear space of great complexity, challenging the usual field quantization techniques.

Having in mind a rigorous quantization of theories of connections and eventually of gravity, methods of calculus in an extension of $\mathcal{A}/\mathcal{G}$ have been developed over the last decade. For a compact gauge group $G$, the extension $\overline{\mathcal{A}/\mathcal{G}}$ is a natural compact measurable space [3], allowing the construction of diffeomorphism invariant measures [4, 5, 6]. An extension $\bar{\mathcal{A}}$ of the space $\mathcal{A}$ of connections was also considered [7]. In this case one still has to divide by the appropriate action of gauge transformations. Besides being equally relevant for integral calculus, the space $\bar{\mathcal{A}}$ is particularly useful for the definition of differential calculus in $\overline{\mathcal{A}/\mathcal{G}}$, fundamental in the construction of quantum observables [8].

These developments rely crucially on the use of Wilson variables (and generalizations), bringing to the foreground the important role of parallel transport defined by certain types of curves. In this contribution we will consider only the case of piecewise analytic curves, for which the formalism was originally introduced, although most of the arguments apply equally well to the more general piecewise smooth case [9, 10]. For both $\bar{\mathcal{A}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ one considers functions on $\mathcal{A}$ of the form

$$\mathcal{A} \ni A \mapsto F\big(h(c_1, A), \ldots, h(c_n, A)\big)\,, \tag{1}$$

where $h(c, A)$ denotes the parallel transport defined by the curve $c$ and $F : G^n \to \mathbb{C}$ is a continuous function. These functions define (overcomplete) coordinates on $\mathcal{A}$. In the case of $\overline{\mathcal{A}/\mathcal{G}}$ only closed curves – loops – are needed, producing gauge invariant functions, or coordinates on $\mathcal{A}/\mathcal{G}$. For compact $G$, the set of all functions (1) is naturally a normed commutative $*$-algebra with identity. Therefore, the norm completion of this $*$-algebra can be identified with the $C^*$-algebra of continuous functions on a compact space called the spectrum of the algebra. This is precisely how the spaces $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}$ were originally introduced: $\overline{\mathcal{A}/\mathcal{G}}$ is the spectrum of the above algebra in the loop case, and $\bar{\mathcal{A}}$ is likewise associated with more general open curves. These spaces are natural completions of $\mathcal{A}/\mathcal{G}$ and $\mathcal{A}$, respectively, and appear as good candidates to replace them in the quantum context [3, 7].

It turns out that both $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}$ can also be seen as projective limits of families of finite dimensional compact manifolds [4, 5, 11]. This projective characterization gives us a great deal of control over the rather complex spaces $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}$, allowing the construction of measures and vector fields on these spaces, starting from corresponding structures on the members of the projective families [4, 5, 7, 8, 11].

For the case of $\overline{\mathcal{A}/\mathcal{G}}$, a distinguished group of equivalence classes of loops, called the hoop group $\mathcal{HG}$ [4], plays an important role: $\overline{\mathcal{A}/\mathcal{G}}$ can in fact be seen as the set of all homomorphisms $\mathcal{HG} \to G$, divided by the action of $G$ [4]. As pointed out by Baez [12], for $\bar{\mathcal{A}}$ a similar role is played by a certain groupoid. In our opinion however, this groupoid associated to open curves has not yet occupied the place it deserves in the literature, possibly due to the fact that groupoids have been introduced in the current mathematical physics literature only recently. Recall that a groupoid is a category such that all arrows are invertible. Therefore, a groupoid generalizes the notion of a group, in the sense that a binary operation with inverse is defined, the difference being that not all pairs of elements can be composed.

In this contribution we consider the projective characterization of $\bar{\mathcal{A}}$ using the language of groupoids from the very beginning and show, using this formalism, that the quotient of $\bar{\mathcal{A}}$ by the action of the gauge group is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$. This new proof, establishing directly the equivalence at the projective limit level, is physically more transparent than the proof one can obtain by combination of results by Ashtekar and Lewandowski [4, 5, 8], Marolf and Mourão [11] and Baez [7].

# 2 Projective characterization of $\bar{\mathcal{A}}$

## 2.1 Edge groupoid

Let $\Sigma$ be an analytic, connected and orientable $d$-manifold. Let $\mathcal{E}$ denote the set of all continuous, oriented and piecewise analytic parametrized curves in $\Sigma$, i.e. continuous maps

$$c : [0, t_1] \cup \ldots \cup [t_{n-1}, 1] \to \Sigma$$

such that the images $c\big(]t_k, t_{k+1}[\big)$ are analytic submanifolds embedded in $\Sigma$. Let $\sigma : \mathcal{E} \to \Sigma$ be the map given by $\sigma(c) = c\big([0,1]\big)$. The maps $s$ (source) and $r$ (range) are defined, respectively, by $s(c) = c(0)$, $r(c) = c(1)$, $c \in \mathcal{E}$. Given two curves $c_1, c_2 \in \mathcal{E}$ such that $s(c_2) = r(c_1)$, let $c_2c_1$ denote the natural composition:

$$(c_2c_1)(t) = \begin{cases} c_1(2t), & \text{for} \quad t \in [0, 1/2] \\ c_2(2t-1), & \text{for} \quad t \in [1/2, 1] \, . \end{cases}$$

Consider also the operation $c \mapsto c^{-1}$ given by $c^{-1}(t) = c(1-t)$. Strictly speaking, the composition of parametrized curves is not associative, since the curves $(c_3c_2)c_1$ and $c_3(c_2c_1)$ are related by a reparametrization. Also, the curve $c^{-1}$ is not the inverse of $c$. Following Isham, Ashtekar and Lewandowski [3, 4] and Baez [12], we describe next an equivalence relation in $\mathcal{E}$ such that the corresponding set of equivalence classes is a well defined groupoid [12], generalizing the hoop group [4].

Let $G$ be a (finite dimensional) connected and compact Lie group and let $P(\Sigma, G)$ be a principal $G$-bundle over $\Sigma$. For simplicity we assume that the bundle is trivial and that a fixed trivialization has been chosen. Let $\mathcal{A}$ be the space of smooth connections on this bundle. The parallel transport associated with a given connection $A \in \mathcal{A}$ and a given curve $c \in \mathcal{E}$ will be denoted by $h(c, A)$.

**Definition 1** *Two elements $c$ and $c'$ of $\mathcal{E}$ are said to be equivalent if*

*(i)* $s(c) = s(c')$, $\quad r(c) = r(c')$

*(ii)* $h(c, A) = h(c', A)$, $\forall A \in \mathcal{A}$.

Two curves related by reparametrization are equivalent and the same is true for curves $c$ and $c'$ that can be written in the form $c = c_2c_1$, $c' = c_2c_3^{-1}c_3c_1$. For noncommutative G, these two conditions are equivalent to $(ii)$ [5, 10], and therefore the equivalence relation is in fact the same for every noncommutative compact Lie group. We will consider noncommutative groups from now on and denote the set of all above defined equivalence classes by $\mathcal{EG}$. It is clear by $(i)$ that the maps $s$ and $r$ are well defined in $\mathcal{EG}$. The map $\sigma$ can still be defined for special elements called edges. By edges we mean elements $e \in \mathcal{EG}$ which are equivalence classes of analytic (in all domain) curves $c : [0,1] \to \Sigma$. It is clear that the images $c_1\big([0,1]\big)$ and $c_2\big([0,1]\big)$ of two equivalent analytic curves coincide, and therefore we define $\sigma(e)$ as being $\sigma(c)$, where $c$ is any analytic curve in the classe of the edge $e$.

We discuss next the natural groupoid structure on the set $\mathcal{EG}$. We will follow the terminology of category theory and refer to elements of $\mathcal{EG}$ as arrows. Given $\gamma, \gamma' \in \mathcal{EG}$ such that $s(\gamma') = r(\gamma)$, one defines the composition $\gamma'\gamma$ by the composition of elements of $\mathcal{E}$. This operation is clearly well defined and is now associative. The points of $\Sigma$ are called objects in this context. Objects are in one-to-one correspondence with identity arrows: given $x \in \Sigma$ the corresponding identity $\mathbf{1}_x \in \mathcal{EG}$ is the equivalence class of $c^{-1}c$, with $c \in \mathcal{E}$ such that $s(c) = x$. If $\gamma$ is the class of $c$ then $\gamma^{-1}$ is the class of $c^{-1}$. It is clear that $\gamma^{-1}\gamma = \mathbf{1}_{s(\gamma)}$ and $\gamma\gamma^{-1} = \mathbf{1}_{r(\gamma)}$. One therefore has a well defined groupoid, whose set of objects is $\Sigma$ and whose set of arrows is $\mathcal{EG}$. As usual, we will use the same notation – $\mathcal{EG}$ – both for the set of arrows and for the groupoid. Notice that every element $\gamma \in \mathcal{EG}$ can be obtained as a composition of edges: the groupoid $\mathcal{EG}$ is generated by edges, although it is not freely generated.

For $x, y \in \Sigma$, let $\mathrm{Hom}\,[x, y]$ be the set of all arrows $\gamma$ such that $s(\gamma) = x$ and $r(\gamma) = y$. It is clear that $\mathrm{Hom}\,[x, x]$ is a group, $\forall x$. Since $\Sigma$ is connected, the groupoid $\mathcal{EG}$ is connected, i.e. $\mathrm{Hom}\,[x, y]$ is a non-empty set $\forall x, y$. In this case, any two groups $\mathrm{Hom}\,[x, x]$ and $\mathrm{Hom}\,[y, y]$ are isomorphic. Let us fix a point $x_0 \in \Sigma$ and consider the group $\mathrm{Hom}\,[x_0, x_0]$. This group is precisely the so-called hoop group $\mathcal{HG}$ [4], whose elements are equivalence classes of piecewise analytic loops. The elements of $\mathrm{Hom}\,[x_0, x_0]$ are called hoops and the identity arrow $\mathbf{1}_{x_0}$ will be called the trivial hoop.

Given that $\mathcal{EG}$ is connected, its elements may be written as compositions of elements of $\mathrm{Hom}\,[x_0, x_0]$ and of an appropriate subset of the set of all arrows:

**Lemma 1** *Suppose that an unique arrow $\gamma_x \in \mathrm{Hom}\,[x_0, x]$ is given for each $x \in \Sigma$, $\gamma_{x_0}$ being the trivial hoop. Then for every $\gamma \in \mathcal{EG}$ there is a uniquely defined $\beta \in \mathrm{Hom}\,[x_0, x_0]$ such that $\gamma = \gamma_{r(\gamma)}\beta\gamma^{-1}_{s(\gamma)}$.*

This result can be obviously adapted for any connected subgroupoid $\Gamma \subset \mathcal{EG}$. The converse of this result is the following lemma, where $\mathrm{Hom}_\Gamma\,[x_0, x_0]$ denotes the subgroup of the hoops that belong to $\Gamma$.

**Lemma 2** *Let $F$ be a subgroup of $\mathrm{Hom}\,[x_0, x_0]$ and $X \subset \Sigma$ be a subset of $\Sigma$ such that $x_0 \in X$. Suppose that an unique arrow $\gamma_x \in \mathrm{Hom}\,[x_0, x]$ is given for each $x \in X$, $\gamma_{x_0}$ being the trivial hoop. Then the set $\Gamma$ of all arrows of the form $\gamma_x\beta\gamma_y^{-1}$, with $\beta \in F$ and $x, y \in X$, is a connected subgroupoid of $\mathcal{EG}$, and the group $\mathrm{Hom}_\Gamma\,[x_0, x_0]$ coincides with $F$.*

### 2.2 $\bar{A}$ as a projective limit

By condition (*ii*) in definition 1, the parallel transport is well defined for any element of $\mathcal{EG}$. To emphasize the algebraic role of connections and to simplify the notation, we will denote by $A(\gamma)$ the parallel transport $h(c, A)$ defined by $A \in \mathcal{A}$ and any curve $c$ in the class $\gamma$. Since the bundle $P(\Sigma, G)$ is trivial,

$A(\gamma)$ defines an element of the group $G$. For every connection $A \in \mathcal{A}$, the map $\mathcal{EG} \to G$ given by

$$\gamma \mapsto A(\gamma) \tag{2}$$

is a groupoid morphism, i.e. $A(\gamma'\gamma) = A(\gamma')A(\gamma)$ and $A(\gamma^{-1}) = A(\gamma)^{-1}$. Thus, there is an injective (but not surjective [3, 4, 7]) map from $\mathcal{A}$ to the set $\mathrm{Hom}\,[\mathcal{EG}, G]$ of all morphisms from $\mathcal{EG}$ to $G$. It turns out that $\mathrm{Hom}\,[\mathcal{EG}, G]$, when equipped with an appropriate topology, is homeomorphic to the space $\bar{\mathcal{A}}$ of generalized connections [11, 5, 12]. This identification can be proved using the fact that $\mathrm{Hom}\,[\mathcal{EG}, G]$ is the projective limit of a projective family labeled by graphs in the manifold $\Sigma$ [2, 8]. In what follows we will rephrase the projective characterization of $\mathrm{Hom}\,[\mathcal{EG}, G]$ using the language of groupoids. Later on we shall consider the action of local gauge transformations on $\mathrm{Hom}\,[\mathcal{EG}, G]$ and show, in the context of projective methods, that the quotient of $\mathrm{Hom}\,[\mathcal{EG}, G]$ by this action is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$.

We start with the set of labels for the projective family leading to $\mathrm{Hom}\,[\mathcal{EG}, G]$, using the notion of independent edges [4].

**Definition 2** *A finite set $\{e_1, \ldots, e_n\}$ of edges is said to be independent if the edges $e_i$ can intersect each other only at the points $s(e_i)$ or $r(e_i)$, $i = 1, \ldots, n$.*

Let $\mathcal{EG}\{e_1, \ldots, e_n\}$ be the subgroupoid of $\mathcal{EG}$ freely generated by the independent set $\{e_1, \ldots, e_n\}$, i.e. the subgroupoid whose objects are all the points $s(e_i)$ and $r(e_i)$, $i = 1, \ldots, n$, and whose arrows are all possible compositions of edges $e_i$ and their inverses. Let $\mathcal{L}$ denote the set of all such subgroupoids. Clearly, the sets $\{e_1, \ldots, e_n\}$ and $\{e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n}\}$, where $e_j^{\epsilon_j} = e_j$ or $e_j^{-1}$, generate the same subgroupoid. Thus, a groupoid $L \in \mathcal{L}$ is uniquely defined by a set $\{\sigma(e_1), \ldots, \sigma(e_n)\}$ of images of a set of independent edges. Notice that the union of the images $\sigma(e_i)$ is a graph in $\Sigma$, thus establishing the relation with the more usual approach using graphs [7, 6, 8]. The set $\mathcal{L}$ is a directed set, i.e. for any given $L, L' \in \mathcal{L}$ there exists $L'' \in \mathcal{L}$ such that both $L$ and $L'$ are subgroupoids of $L''$, $L'' \supset L$ and $L'' \supset L'$. This follows from the crucial fact that for every finitely generated subgroupoid $\Gamma \subset \mathcal{EG}$ there is an element $L \in \mathcal{L}$ such that $\Gamma \subset L$, which can be easily proved in the piecewise analytic case [4].

Let us now consider the projective family. For each $L \in \mathcal{L}$, let $\mathcal{A}_L := \mathrm{Hom}\,[L, G]$ be the set of all morphisms from the groupoid $L$ to the group $G$. The family of spaces $\mathcal{A}_L,\ L \in \mathcal{L}$, is a so-called compact Hausdorff projective family [5], meaning that each of the spaces $\mathcal{A}_L$ is a compact Hausdorff space and that $\forall L, L' \in \mathcal{L}$ such that $L' \supset L$ there exists a surjective and continuous projection $p_{L,L'} : \mathcal{A}_{L'} \to \mathcal{A}_L$ satisfying

$$p_{L,L''} = p_{L,L'} \circ p_{L',L''},\ \forall L'' \supset L' \supset L\,. \tag{3}$$

There is a well defined notion of limit of the family of spaces $\mathcal{A}_L$ – the projective limit – which is also a compact Hausdorff space. We discuss next some aspects of this construction.

Let $L = \mathcal{EG}\{e_1, \dots, e_n\}$ be an element of $\mathcal{L}$. Since the morphisms $L \to G$ are determined by the images of the generators, one gets a bijection $\rho_{e_1,\dots,e_n}$ : $\mathcal{A}_L \to G^n$,

$$\mathcal{A}_L \ni \bar{A} \mapsto \big(\bar{A}(e_1), \dots, \bar{A}(e_n)\big) \in G^n\,. \tag{4}$$

Through this identification with $G^n$, the space $\mathcal{A}_L$ can be seen as a compact Hausdorff space. For $L' \supset L$ the projection $p_{L,L'} : \mathcal{A}_{L'} \to \mathcal{A}_L$ is defined as the map sending each element of $\mathcal{A}_{L'}$ to its restriction to $L$. Using the maps (4) it is not difficult to see that the projections $p_{L,L'}$ are surjective and continuous [5, 13].

The projective limit of the family $\{\mathcal{A}_L, p_{L,L'}\}$ is the subset $\mathcal{A}_\infty$ of the cartesian product $\mathsf{X}_{L\in\mathcal{L}}\mathcal{A}_L$ of the elements $(A_L)_{L\in\mathcal{L}}$ satisfying the consistency conditions:

$$p_{L,L'}A_{L'} = A_L\,, \quad \forall\, L' \supset L\,. \tag{5}$$

The cartesian product is a compact Hausdorff space with respect to the Tychonov product topology. Given the continuity of the projections $p_{L,L'}$, the projective limit $\mathcal{A}_\infty$ is a closed subset [11, 5] and therefore is also compact Hausdorff. The induced topology in $\mathcal{A}_\infty$ is the weakest topology such that all the following projections are continuous:

$$\begin{array}{rccc} p_L : & \mathcal{A}_\infty & \to & \mathcal{A}_L \\ & (A_L)_{L\in\mathcal{L}} & \mapsto & A_L\,. \end{array} \tag{6}$$

The proof that the projective limit $\mathcal{A}_\infty$ coincides with the set of all groupoid morphisms $\mathrm{Hom}\,[\mathcal{EG}, G]$ follows essentially the same steps as the proof of the well known fact that the algebraic dual of any vector space is a projective limit. In what follows we will identify $\mathcal{A}_\infty$ with $\mathrm{Hom}\,[\mathcal{EG}, G]$. For simplicity, we will refer to the induced topology on $\mathrm{Hom}\,[\mathcal{EG}, G]$ as the Tychonov topology.

# 3 Equivalence of the projective characterizations of $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$

In this section we will study the relation between the space of generalized connections considered above and the space $\overline{\mathcal{A}/\mathcal{G}}$ of generalized connections modulo gauge transformations, from the point of view of projective techniques. The gauge transformations act naturally on $\mathrm{Hom}\,[\mathcal{EG}, G]$ and, as expected, the quotient of $\mathrm{Hom}\,[\mathcal{EG}, G]$ by this action is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$. The proof presented here complements previous results [4, 5, 11, 7, 8] and clarifies the relation between the two spaces. The introduction of the groupoid $\mathcal{EG}$ plays a relevant role in this result.

We start with a brief review of the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$ [4, 5, 11]. In this case the projective family is labeled by certain "tame" subgroups of the hoop group $\mathcal{HG} \equiv \mathrm{Hom}\,[x_0, x_0]$, which are subgroups freely generated by finite sets of independent hoops. We will denote the family of such subgroups by $\mathcal{S}_\mathcal{H}$. For each $S \in \mathcal{S}_\mathcal{H}$ one considers the set $\chi_S$ of all homomorphisms $S \to G$, $\chi_S := \mathrm{Hom}[S, G]$. The family $\{\chi_S\}_{S\in\mathcal{S}_\mathcal{H}}$ is a compact Hausdorff projective family, whose projective limit is Hom $[\mathcal{HG}, G]$, the set of all homomorphisms $\mathcal{HG} \to G$ [5]. The space Hom $[\mathcal{HG}, G]$ is equiped with a Tychonov-like topology, namely the weakest topology such that all the natural projections

$$p_S : \mathrm{Hom}[\mathcal{HG}, G] \to \chi_S\,, \quad S \in \mathcal{S}_{\mathcal{H}}\,, \tag{7}$$

are continuous. The group $G$ acts continuously on Hom $[\mathcal{HG}, G]$ [5]:

$$\mathrm{Hom}\,[\mathcal{HG}, G] \times G \ni (H, g) \mapsto H_g \ : \ H_g(\beta) = g^{-1}H(\beta)g,\ \forall \beta \in \mathcal{HG}. \tag{8}$$

This action corresponds to the non-trivial part of the action of the group of generalized local gauge transformations (see below). It is a well established fact that the quotient space Hom $[\mathcal{HG}, G]/G$ is homeomorphic to $\overline{\mathcal{A}/\mathcal{G}}$ [11, 5].

Let us consider now the corresponding action of local gauge transformations on generalized connections. The group of local gauge transformations associated with the structure group $G$ is the group $\mathcal{G}$ of all smooth maps $\mathrm{g} : \Sigma \to G$, acting on smooth connections as follows:

$$\mathcal{A} \ni A \mapsto \mathrm{g}^{-1}A\mathrm{g} + \mathrm{g}^{-1}d\mathrm{g}\,,$$

where $d$ denotes the exterior derivative. The corresponding action on parallel transports $A(\gamma)$ defined by $A \in \mathcal{A}$ and $\gamma \in \mathcal{EG}$ is given by

$$A(\gamma) \mapsto \mathrm{g}(x_2)^{-1}A(\gamma)\mathrm{g}(x_1) \ \ \mathrm{g} \in \mathcal{G}\,, \tag{9}$$

where $x_1 = s(\gamma)$, $x_2 = r(\gamma)$. Let us consider the extension $\bar{\mathcal{G}}$ of $\mathcal{G}$,

$$\bar{\mathcal{G}} = \mathrm{Map}[\Sigma, G] = G^{\Sigma} \cong \mathsf{X}_{x \in \Sigma} G_x\,, \tag{10}$$

of all maps $\mathrm{g} : \Sigma \to G$, not necessarily smooth or even continuous. This group $\bar{\mathcal{G}}$ of "generalized local gauge transformations" acts naturally on the space of generalized connections Hom $[\mathcal{EG}, G]$,

$$\mathrm{Hom}\,[\mathcal{EG}, G] \times \bar{\mathcal{G}} \ni (\bar{A}, \mathrm{g}) \mapsto \bar{A}_{\mathrm{g}} \in \mathrm{Hom}\,[\mathcal{EG}, G] \tag{11}$$

where

$$\bar{A}_{\mathrm{g}}(\gamma) = \mathrm{g}\big(r(\gamma)\big)^{-1}\bar{A}(\gamma)\mathrm{g}\big(s(\gamma)\big),\ \forall \gamma \in \mathcal{EG}\,, \tag{12}$$

generalizing (9). It is natural to consider the quotient of Hom$[\mathcal{EG}, G]$ by the action of $\bar{\mathcal{G}}$, since Hom$[\mathcal{EG}, G]$ is also made of all the morphisms $\mathcal{EG} \to G$, without any continuity condition. The group $\bar{\mathcal{G}}$ is compact Hausdorff and its action is continuous [5, 8]. Therefore Hom$[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is also a compact Hausdorff space.

Let us consider the compact space $\bar{\mathcal{A}}$ introduced by Baez as the spectrum of a $C^*$-algebra, obtained by completion of a $*$-algebra of functions in $\mathcal{A}$ [7]. The original $C^*$-algebra can then be identified with the algebra $C(\bar{\mathcal{A}})$ of continuous functions in $\bar{\mathcal{A}}$. The group of local gauge transformations acts naturally on $C(\bar{\mathcal{A}})$ and the subspace $C^{\mathcal{G}}(\bar{\mathcal{A}}) \subset C(\bar{\mathcal{A}})$ of gauge invariant functions is also a commutative $C^*$-algebra with identity [7], whose spectrum we will denote by $\bar{\mathcal{A}}/\bar{\mathcal{G}}$.

One therefore has four extensions of the classical configuration space $\mathcal{A}/\mathcal{G}$, namely $\overline{\mathcal{A}/\mathcal{G}}$, $\bar{\mathcal{A}}/\bar{\mathcal{G}}$, $\mathrm{Hom}\,[\mathcal{HG}, G]/G$ and $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}}$. The first two spaces are tied to the $C^*$-algebra formalism whereas the last two appear in the context of projective methods. As expected, all these spaces are naturally homeomorphic. Let us consider the following diagram

$$\begin{array}{ccc} \overline{\mathcal{A}/\mathcal{G}} & \longleftrightarrow & \mathrm{Hom}\,[\mathcal{HG}, G]/G \\ \updownarrow & & \\ \bar{\mathcal{A}}/\bar{\mathcal{G}} & \longleftrightarrow & \mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}} \end{array}$$

The correspondence between $\overline{\mathcal{A}/\mathcal{G}}$ and $\mathrm{Hom}\,[\mathcal{HG}, G]/G$ was established by Marolf and Mourão [11]. A generalization of this result produces a homeomorphism between $\bar{\mathcal{A}}$ and $\mathrm{Hom}\,[\mathcal{EG}, G]$ [5], which is easily seen to be equivariant, leading to a homeomorphism between $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ and $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}}$ [8]. The correspondence between $\overline{\mathcal{A}/\mathcal{G}}$ and $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ follows from results by Baez [7]. In the remaining of this section we will show directly that $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is homeomorphic to $\mathrm{Hom}[\mathcal{HG}, G]/G$. The relevance of this new proof of a known result lies in the clear relation established between $\mathrm{Hom}\,[\mathcal{EG}, G]$ ($\cong \bar{\mathcal{A}}$) and $\mathrm{Hom}\,[\mathcal{HG}, G]/G$ ($\cong \overline{\mathcal{A}/\mathcal{G}}$), without having to rely on the characterization of these spaces as spectra of $C^*$-algebras.

Since $\mathcal{HG} \equiv \mathrm{Hom}\,[x_0, x_0]$ is a subgroup of the groupoid $\mathcal{EG}$, a projection from $\mathrm{Hom}\,[\mathcal{EG}, G]$ to $\mathrm{Hom}\,[\mathcal{HG}, G]$, given by the restriction of elements of $\mathrm{Hom}\,[\mathcal{EG}, G]$ to the group $\mathcal{HG}$, is naturally defined. We will show that this projection is surjective and equivariant with respect to the actions of $\bar{\mathcal{G}}$ on $\mathrm{Hom}\,[\mathcal{EG}, G]$ and $\mathrm{Hom}\,[\mathcal{HG}, G]$, thus defining a map $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}} \to \mathrm{Hom}\,[\mathcal{HG}, G]/G$ which is in fact a bijection. We will also present the more relevant elements of the proof that the latter map and its inverse are continuous. For this we will need some lemmas, whose proofs will appear elsewhere [13].

We will start by showing that $\mathrm{Hom}\,[\mathcal{EG}, G]$ is homeomorphic to $\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$, where $\bar{\mathcal{G}}_{x_0}$ is the subgroup of $\bar{\mathcal{G}}$ (10) of the elements $\mathrm{g}$ such that $\mathrm{g}(x_0) = \mathbf{1}$. Let us fix a unique edge $e_x \in \mathrm{Hom}\,[x_0, x]$ for each $x \in \Sigma$, $e_{x_0}$ being the trivial hoop. Let us denote this set of edges by $\Lambda = \{e_x,\ x \in \Sigma\}$. Consider the map

$$\Theta_\Lambda : \mathrm{Hom}\,[\mathcal{EG}, G] \to \mathrm{Hom}[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \tag{13}$$

where $\bar{A} \in \mathrm{Hom}\,[\mathcal{EG}, G]$ is mapped to $(H, \mathrm{g}) \in \mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ such that

$$H(\beta) = \bar{A}(\beta), \quad \forall \beta \in \mathcal{HG} \tag{14}$$

and

$$\mathrm{g}(x) = \bar{A}\big(e_x\big), \quad \forall x \in \Sigma\,. \tag{15}$$

Consider also the natural action of $\bar{\mathcal{G}}$ on $\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ given by

$$\big(\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}\big) \times \bar{\mathcal{G}} \ni \big((H, \mathrm{g}), \mathrm{g}'\big) \mapsto (H_{\mathrm{g}'}, \mathrm{g}_{\mathrm{g}'})\,, \tag{16}$$

where

$$H_{\mathrm{g}'}(\beta) = \mathrm{g}'(x_0)^{-1} H(\beta) \mathrm{g}'(x_0), \quad \forall \beta \in \mathcal{HG} \tag{17}$$

and

$$\mathrm{g}_{\mathrm{g}'}(x) = \mathrm{g}'(x)^{-1} \mathrm{g}(x) \mathrm{g}'(x_0), \quad \forall x \in \Sigma\,. \tag{18}$$

**Theorem 1** *For any choice of the set $\Lambda$, the map $\Theta_\Lambda$ is a homeomorphism, equivariant with respect to the action of $\bar{\mathcal{G}}$.*

It is fairly easy to see that $\Theta_\Lambda$ is bijective and equivariant: for a given $\Lambda$, the map $\Theta_\Lambda$ is clearly well defined and its inverse is given by $(H, \mathrm{g}) \mapsto \bar{A}$ where

$$\bar{A}(\gamma) = \mathrm{g}\big(r(\gamma)\big) H\Big(e^{-1}_{r(\gamma)} \gamma\, e_{s(\gamma)}\Big) \mathrm{g}\big(s(\gamma)\big)^{-1}, \quad \forall \gamma \in \mathcal{EG}\,. \tag{19}$$

It is clear that $\Theta_\Lambda$ is equivariant with respect to the action of $\bar{\mathcal{G}}$ on $\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ (17, 18) and on $\mathrm{Hom}\,[\mathcal{EG}, G]$ (11, 12). It remains to be shown that $\Theta_\Lambda$ is a homeomorphism. Recall that the topologies of $\mathrm{Hom}\,[\mathcal{HG}, G]$ and $\mathrm{Hom}\,[\mathcal{EG}, G]$ are defined by the projective families $\{\chi_S\}_{S \in \mathcal{S}_\mathcal{H}}$ and $\{\mathcal{A}_L\}_{L \in \mathcal{L}}$ considered previously.

Given $S \in \mathcal{S}_\mathcal{H}$ and $x \in \Sigma$, let $P_S$ and $\pi_x$, respectively, be the projections from $\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ to $\chi_S$ and $G_x$ (the copy of $G$ associated with the point $x$). Recall that the topology of $\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}$ is the weakest topology such that all the maps $P_S$ and $\pi_x$ are continuous. So, $\Theta_\Lambda$ is continuous if and only if the maps $P_S \circ \Theta_\Lambda$ and $\pi_x \circ \Theta_\Lambda$ are continuous, $\forall S \in \mathcal{S}_\mathcal{H}$ and $\forall x \in \Sigma$. Likewise, $\Theta_\Lambda^{-1}$ is continuous if and only if all the maps $p_L \circ \Theta_\Lambda^{-1} : \mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0} \to \mathcal{A}_L$ are continuous, where the projections $p_L : \mathrm{Hom}\,[\mathcal{EG}, G] \to \mathcal{A}_L$ are defined in (6).

It is straightforward to show that the maps $\pi_x \circ \Theta_\Lambda$ are continuous: given $x \in \Sigma$, one just has to consider the subgroupoid $L = \mathcal{EG}\big\{e_x\big\}$ generated by the edge $e_x \in \Lambda$ and the homeomorphism (4) $\rho_{e_x} : \mathcal{A}_L \to G$. It is clear that $\pi_x \circ \Theta_\Lambda$ coincides with $\rho_{e_x} \circ p_L$, being therefore continuous.

On the other hand, to show that $P_S \circ \Theta_\Lambda$ and $p_L \circ \Theta_\Lambda^{-1}$ are continuous one needs to consider explicitly the relation between the spaces $\mathcal{A}_L$ and $\chi_S$, $L \in \mathcal{L}$, $S \in \mathcal{S}_\mathcal{H}$ [13].

**Lemma 3** *For every $S \in \mathcal{S}_\mathcal{H}$ there exists a connected subgroupoid $L \in \mathcal{L}$ such that $S$ is a subgroup of $L$. The projection*

$$p_{S,L} : \mathcal{A}_L \to \chi_S \tag{20}$$

*defined by the restriction of elements of $\mathcal{A}_L$ to $S$ is continuous and satisfies*

$$P_S \circ \Theta_\Lambda = p_{S,L} \circ p_L\,, \quad \forall \Lambda. \tag{21}$$

The continuity of the maps $P_S \circ \Theta_\Lambda$, $\forall S \in \mathcal{S}_\mathcal{H}$, follows immediately from lemma 3. To show that the maps $p_L \circ \Theta_\Lambda^{-1}$ are continuous one needs the converse of lemma 3. Recall that for a given subgroupoid $\Gamma \subset \mathcal{EG}$, $\mathrm{Obj}\,\Gamma$ denotes the set of objects of $\Gamma$ (the set of all points of $\Sigma$ which are range or source for some arrow in $\Gamma$) and that $\mathrm{Hom}\,[\Gamma, G]$ stands for the set of all morphisms $\Gamma \to G$. We will also denote by $\Pi_\Gamma$ the natural projection from $\bar{\mathcal{G}}_{x_0}$ to the subgroup $\bar{\mathcal{G}}_{x_0}(\Gamma)$ of all maps $\mathrm{Obj}\,\Gamma \to G$ such that $\mathrm{g}(x_0) = \mathbf{1}$.

**Lemma 4** *For every $L \in \mathcal{L}$ there exists $S \in \mathcal{S}_{\mathcal{H}}$ and a connected subgroupoid $\Gamma \subset \mathcal{EG}$, with $\mathrm{Obj}\,\Gamma = \mathrm{Obj}\,L \,\dot{\cup}\, \{x_0\}$, such that $L \subset \Gamma$ and $\mathrm{Hom}_{\Gamma}[x_0, x_0] = S$. The natural projection from $\mathrm{Hom}\,[\Gamma, G]$ to $\mathcal{A}_L$ defines a map*

$$p_{L,S} : \chi_S \times \bar{\mathcal{G}}_{x_0}(\Gamma) \to \mathcal{A}_L \tag{22}$$

*which is continuous and such that for an appropriate choice of $\Lambda$ one has*

$$p_L \circ \Theta_{\Lambda}^{-1} = p_{L,S} \circ (p_S \times \Pi_{\Gamma})\,. \tag{23}$$

Given that the projections $p_S : \mathrm{Hom}\,[\mathcal{HG}, G] \to \chi_S$ and $\Pi_{\Gamma} : \bar{\mathcal{G}}_{x_0} \to \bar{\mathcal{G}}_{x_0}(\Gamma)$ are continuous, lemma 4 shows that for every fixed $L \in \mathcal{L}$ there exists a $\Lambda$ such that $p_L \circ \Theta_{\Lambda}^{-1}$ is continuous, which still does not prove that all the maps $p_L \circ \Theta_{\Lambda}^{-1}$ are continuous for a given $\Lambda$. Notice however that the map $\Theta_{\Lambda} \circ \Theta_{\Lambda'}^{-1}$ is a homeomorphism for any $\Lambda$ and $\Lambda'$ [13], from what follows immediately that

**Lemma 5** *The continuity of $p_L \circ \Theta_{\Lambda}^{-1}$ is equivalent to the continuity of $p_L \circ \Theta_{\Lambda'}^{-1}$, for any other values of $\Lambda'$.*

This lemma, together with lemma 4, shows that, for a given $\Lambda$, all the maps $p_L \circ \Theta_{\Lambda}^{-1}$, $L \in \mathcal{L}$, are continuous, which concludes the proof of theorem 1.

The identification of $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}}$ with $\mathrm{Hom}\,[\mathcal{HG}, G]/G$ now follows easily. Consider a fixed $\Lambda$. Since $\Theta_{\Lambda}$ is a homeomorphism equivariant with respect to the continuous action of $\bar{\mathcal{G}}$, we conclude that $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}}$ is homeomorphic to $\big(\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}\big)/\bar{\mathcal{G}}$. On the other hand it is clear that

$$\begin{aligned} \big(\mathrm{Hom}\,[\mathcal{HG}, G] \times \bar{\mathcal{G}}_{x_0}\big)/\bar{\mathcal{G}} &= \big(\mathrm{Hom}\,[\mathcal{HG}, G]/G\big) \times \big(\bar{\mathcal{G}}_{x_0}/\bar{\mathcal{G}}_{x_0}\big) \cong \\ &\cong \mathrm{Hom}\,[\mathcal{HG}, G]/G\,. \end{aligned} \tag{24}$$

Thus, as a corollary of theorem 1 one gets that

**Theorem 2** *The spaces $\mathrm{Hom}\,[\mathcal{EG}, G]/\bar{\mathcal{G}}$ and $\mathrm{Hom}\,[\mathcal{HG}, G]/G$ are homeomorphic.*

## Acknowledgements

I would like to thank José Mourão, Paulo Sá and Thomas Thiemann, for encouragement and helpful discussions. This work was supported in part by CENTRA/UAlg, PRAXIS 2/2.1/FIS/286/94 and CERN/P/FIS/15196/1999.

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